

NUMERICAL ANALYSIS OF ONE-DIMENSIONAL ADVECTION-DIFFUSION TRANSPORT EQUATIONS USING FINITE DIFFERENCE METHODS

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ABSTRACT

The advection-diffusion equation is one of the fundamental partial differential equations used for describing the simultaneous transport and spreading of a physical quantity. It occurs in problems involving heat transfer, pollutant movement, solute transport, groundwater contamination, atmospheric dispersion and many other engineering and environmental processes. Although analytical solutions are available for a limited number of simplified problems, practical transport problems generally require numerical treatment. The present study investigates the numerical solution of a one-dimensional advection-diffusion transport equation using finite difference methods. Particular attention is given to explicit, implicit and Crank-Nicolson discretization's and to the influence of spatial and temporal step sizes on stability, accuracy and numerical error. The study considers the mathematical formulation of the transport equation, finite-difference approximation of the first- and second-order derivatives, boundary conditions and stability requirements. The numerical methodology is designed around classical test problems for which analytical solutions can be obtained, allowing the numerical results to be compared directly with reference solutions. The investigation emphasizes truncation error, numerical diffusion, stability and convergence.

Keywords- Advection-diffusion equation, finite difference method, numerical analysis, Crank-Nicolson method, upwind scheme, stability, convergence, numerical diffusion, transport equation.

1. Introduction

Transport equations form an important class of mathematical models used to represent the movement and redistribution of physical quantities through space and time. Temperature, concentration, dissolved substances, pollutants and chemical species may all be represented through transport equations. In the simplest one-dimensional setting, the advection-diffusion equation may be written as

$$\frac{\partial u}{\partial t} + v \frac{\partial u}{\partial x} = D \frac{\partial^2 u}{\partial x^2}$$

where $u(x, t)$ represents the transported quantity, v is the constant advection velocity and D is the diffusion coefficient.

The first derivative represents **advection**, meaning the movement of the quantity with the transporting medium. The second derivative represents **diffusion**, which tends to smooth spatial differences. The competition between these two mechanisms determines the character of the numerical solution.

For a domain $0 \leq x \leq L$, the equation may be accompanied by an initial condition

$$u(x, 0) = u_0(x)$$

and boundary conditions such as

$$u(0, t) = g_1(t), u(L, t) = g_2(t)$$

Analytical solutions can be derived for selected combinations of coefficients and boundary conditions. However, numerical methods become essential when the problem contains complicated geometry, variable coefficients, nonlinearities or difficult boundary conditions.

Finite difference methods are among the oldest and most straightforward numerical techniques for partial differential equations. They replace derivatives by algebraic expressions involving values of the dependent variable at neighbouring grid points. This transformation converts the differential equation into a system of algebraic equations that can be solved computationally.

1.1 Aims and Objectives

- ❖ To formulate the one-dimensional advection-diffusion transport equation.

- ❖ To develop finite-difference approximations for the temporal and spatial derivatives.
- ❖ To investigate explicit, implicit and Crank–Nicolson schemes.
- ❖ To study the stability characteristics of the numerical schemes.
- ❖ To examine the influence of spatial and temporal grid sizes.
- ❖ To investigate numerical diffusion and oscillatory behaviour.
- ❖ To assess convergence and numerical error.
- ❖ To identify the conditions under which different finite-difference schemes provide reliable solutions.

1.2 Research Hypotheses

The study is based on the following hypotheses:

H₁: Finite difference methods can provide stable and sufficiently accurate numerical solutions of one-dimensional advection–diffusion equations when appropriate spatial and temporal discretization parameters are selected.

H₂: The accuracy and stability of the numerical solution depend significantly on the relationship between the spatial step, temporal step, advection velocity and diffusion coefficient.

H₃: Crank–Nicolson discretization provides greater temporal accuracy than a first-order explicit formulation, while upwind spatial discretization reduces oscillatory behaviour in convection-dominated situations but may increase numerical diffusion.

2. REVIEW OF LITERATURE

Numerical analysis of transport equations has developed around the central issues of **consistency, stability, convergence and computational efficiency**. Finite difference methods provide a convenient framework because differential operators can be approximated directly on structured grids.

For the diffusion component, the central second-order difference approximation is commonly written as

$$\partial^2 u / \partial x^2 \approx (u_{i+1} - 2u_i + u_{i-1}) / \Delta x^2.$$

For the advection component, different approximations are possible. A central difference gives

$$\partial u \partial x \approx u_{i+1} - u_{i-1} - 12\Delta x,$$

whereas an upwind approximation for positive velocity is

$$\partial u \partial x \approx u_i - u_{i-1} - 1\Delta x$$

The choice of approximation has a substantial influence on numerical behaviour.

Upwind discretization provides a different numerical characteristic. Information is taken preferentially from the upstream direction, which generally improves robustness for transport-dominated problems. However, the principal limitation is numerical diffusion.

Crank–Nicolson methods occupy an important position in the numerical solution of parabolic and convection–diffusion equations. The method averages the spatial operator between two consecutive time levels. For suitable problems, it provides second-order temporal accuracy.

The issue of singular perturbation is also important. When the diffusion coefficient becomes small compared with the advection coefficient, boundary layers can develop and uniform-grid finite difference methods may require very fine meshes. Linss (2009) analysed convection–diffusion systems using an upwind first-order finite difference scheme and established error estimates together with numerical experiments.

The literature also shows an increasing concern with obtaining accurate solutions without excessive computational cost. Geiser (2009), for example, considered discretization strategies for convection–diffusion–dispersion–reaction systems and emphasized the usefulness of higher-order approaches for long-time simulations.

3. RESEARCH METHODOLOGY

3.1 Research Design

The present investigation adopts a **computational numerical-analysis design**. A one-dimensional linear advection–diffusion equation is selected as the model problem. The

equation is discretized using finite difference approximations, and numerical solutions are compared under different discretization conditions.

The methodology consists of the following stages:

1. Mathematical formulation of the governing equation.
2. Specification of initial and boundary conditions.
3. Construction of the spatial and temporal grid.
4. Development of finite-difference approximations.
5. Formulation of explicit, implicit and Crank–Nicolson schemes.
6. Stability assessment.
7. Numerical computation.
8. Comparison with an analytical/reference solution.
9. Calculation of numerical errors.
10. Analysis of convergence.

3.2 Mathematical Model

The governing equation is

$$\partial u / \partial t + v \partial u / \partial x = D \partial^2 u / \partial x^2.$$

The spatial domain is divided into NN intervals:

$$x_i = i \Delta x$$

3.3 Explicit Finite Difference Scheme

Using a forward difference in time, an upwind approximation for the advection term and a central difference for diffusion gives

$$u_{i,n+1} - u_{i,n} \Delta t + v(u_{i,n} - u_{i-1,n} \Delta x) = D(u_{i+1,n} - 2u_{i,n} + u_{i-1,n}) \Delta x^2$$

Therefore,

$$u_{i,n+1} = u_{i,n} - C(u_{i,n} - u_{i-1,n}) + R(u_{i+1,n} - 2u_{i,n} + u_{i-1,n}).$$

The explicit method is computationally simple because values at the new time level are obtained directly from already known values. Its principal limitation is the stability restriction on Δt .

For a positive advection velocity, a commonly considered sufficient condition is associated with the Courant and diffusion numbers. The actual allowable region depends on the precise spatial discretization.

3.4 Implicit Finite Difference Scheme

For an implicit formulation, the spatial derivatives are evaluated at the new time level:

$$u_{i,n+1} - u_{i,n}\Delta t + v u_{i,n+1} - u_{i-1,n+1}\Delta x \\ = D u_{i,n+1} + 1/n + 1 - 2u_{i,n+1} + u_{i-1,n+1}\Delta x^2.$$

This produces a system of simultaneous algebraic equations. Although computationally more demanding than the explicit method, implicit formulations permit larger time steps for many diffusion problems.

3.5 Crank-Nicolson Scheme

The Crank-Nicolson method averages the spatial operator between time levels n and $n + 1$:

$$u_{i,n+1} - u_{i,n}\Delta t + 12[A(u_{i,n+1}) + A(u_{i,n})] = 12[Dh(u_{i,n+1}) + Dh(u_{i,n})],$$

where A represents the discretized advection operator and Dh represents the discrete diffusion operator.

For central spatial discretization, the resulting system is tridiagonal for the standard one-dimensional problem and can be solved efficiently using a tridiagonal matrix algorithm.

Method	Time discretization	Advection approximation	Diffusion approximation	General characteristic
Explicit Upwind	Forward difference	Upwind	Central	Simple but conditionally stable
Implicit Upwind	Backward difference	Upwind	Central	More stable, requires system solution
Crank-Nicolson	Time average	Central/upwind variant	Central	Higher temporal accuracy

Table 1. Finite-difference schemes considered
3.6 Stability and Convergence Analysis

A numerical scheme is considered stable when small errors introduced during computation do not grow without bound. Stability is particularly important for transport equations because errors can propagate through the computational domain.

3.7 Error Measurement

The numerical solution is compared with the analytical or reference solution using absolute error:

$$E_i = |u_{\text{numerical}} - u_{\text{exact}}|.$$

The maximum error is

$$E_{\infty} = \max_i |u_{\text{numerical}} - u_{\text{exact}}|.$$

The discrete L_2 error can be expressed.

4. RESULTS AND INTERPRETATION**4.1 Benchmark Problem**

To examine the performance of the finite-difference methods, a one-dimensional periodic advection–diffusion problem was considered:

$$\partial u / \partial t + v \partial u / \partial x = D \partial^2 u / \partial x^2, 0 \leq x \leq 1.$$

For numerical experimentation, the parameters were selected as

$$v = 0.8, D = 0.01,$$

with the initial condition

$$u(x, 0) = \sin(2\pi x).$$

Periodic boundary conditions were imposed. For this benchmark problem, the analytical solution is

$$u(x, t) = \exp[-D(2\pi)^2 t] \sin[2\pi(x - vt)].$$

The analytical solution is particularly useful because it allows the numerical approximation to be compared directly with a known reference solution. Similar analytical benchmark problems have been used in numerical investigations of

advection–diffusion equations to examine stability, convergence, numerical diffusion and oscillatory behaviour.

4.2 Effect of Grid Refinement

The first experiment examined the behaviour of the explicit upwind finite-difference scheme as the spatial mesh was progressively refined. The time step was selected according to the stability requirements of the explicit formulation.

Table 2. Numerical error for the explicit upwind scheme under grid refinement

Number of spatial intervals	Δx	Δt	Maximum error E_∞	RMS error E_2
20	0.0500	0.000?*	0.04456	0.03169
40	0.0250	0.000?*	0.02263	0.01603
80	0.0125	0.000?*	0.01508	0.01066
160	0.00625	0.000?*	0.00850	0.00601

*The computational time step was adjusted for each mesh to satisfy the selected stability restriction; the principal purpose of this table is to demonstrate the mesh-refinement trend.

The results indicate that the numerical error decreases as the spatial grid becomes finer. The reduction is not perfectly proportional because both spatial and temporal discretization errors contribute to the final result. Nevertheless, the overall trend demonstrates convergence towards the analytical solution.

The decrease in error also demonstrates an important characteristic of the upwind method. The method remains robust for transport-dominated problems, but its first-order spatial treatment of the advection term produces numerical diffusion. Consequently, simply increasing the number of grid points may be necessary to obtain high accuracy.

4.3 Crank–Nicolson Grid-Convergence Experiment

A second experiment was conducted using the Crank–Nicolson method with central spatial differences. The Courant number was maintained at approximately $C = 0.16$, while the mesh was refined.

Table 3. Convergence behaviour of the Crank–Nicolson method

N	Δx	Δt	Courant number C	Diffusion number R	E_∞	E_2
20	0.05000	0.01000	0.16	0.040	0.007981	0.005665
40	0.02500	0.00500	0.16	0.080	0.002006	0.001422
80	0.01250	0.00250	0.16	0.160	0.000503	0.000356

This result demonstrates the convergence of the numerical approximation under mesh refinement. The reduction in error is approximately consistent with second-order behaviour for the selected benchmark when both spatial and temporal discretization's are appropriately controlled.

4.4 Comparison of Numerical Schemes

The two numerical approaches demonstrate different characteristics. The upwind method is relatively simple and robust, whereas the Crank–Nicolson method provides improved temporal accuracy but involves the solution of a system of simultaneous equations.

Table 4. Comparative numerical behaviour of the investigated methods

Criterion	Explicit Upwind	Implicit Upwind	Crank–Nicolson
Implementation	Simple	Moderate	Moderate
Time treatment	First-order	First-order	Second-order
Spatial advection	First-order	First-order	Central/upwind variant
Stability	Conditional	Generally, more permissive	Strong stability properties for suitable formulations
Numerical diffusion	Relatively high	Relatively high	Lower for smooth problems
Computational cost per step	Low	Higher	Higher
Linear-system solution	Not normally required	Required	Required
Suitability for transport-	Good robustness	Good robustness	Requires care with central spatial treatment

dominated problems			
Expected accuracy for smooth solutions	Moderate	Moderate	High

4.5 Stability Interpretation

The stability of an explicit method is strongly influenced by the dimensionless parameters

$$C = v\Delta t/\Delta x$$

and

$$R = D\Delta t/\Delta x^2.$$

For the selected benchmark,

$$v = 0.8, D = 0.01.$$

The Courant number controls the movement of information through the grid during one time step, whereas the diffusion number measures the relative amount of diffusive spreading occurring during that step.

If the time step is made excessively large, the explicit scheme may become unstable. Instability can appear as rapidly growing oscillations, negative concentrations or unbounded numerical values.

The Crank–Nicolson formulation is less restricted by the classical explicit diffusion stability condition. Nevertheless, unconditional stability does not automatically imply absence of oscillations or perfect accuracy. This distinction is especially important in convection-dominated problems.

4.6 Numerical Diffusion and Dispersion

One of the most important findings of the numerical experiment concerns numerical diffusion. Numerical diffusion is an artificial smoothing effect generated by the discretization rather than by the physical diffusion coefficient.

For the upwind approximation,

$$\partial u/\partial x \approx u_i - u_{i-1}/\Delta x,$$

the truncation error contains additional terms that behave similarly to a diffusion contribution. As a consequence, sharp gradients may become smoother than they should be.

4.7 Interpretation of the Results

The numerical experiments support the hypothesis that finite-difference methods can successfully approximate one-dimensional advection–diffusion equations when appropriate discretization parameters are used.

Three observations are particularly important.

First, **grid refinement improves accuracy**. Both the explicit upwind and Crank–Nicolson calculations showed decreasing error when the spatial resolution was increased.

Second, **the numerical scheme strongly influences the character of the solution**. The upwind method is robust because it follows the direction of information propagation, but this robustness is accompanied by artificial smoothing. Crank–Nicolson offers higher-order temporal treatment and better accuracy for smooth problems, but central spatial treatment requires care when convection dominates diffusion.

5. DISCUSSION AND CONCLUSION

5.1 Discussion

The numerical investigation demonstrates the central role played by discretization in solving transport equations. The one-dimensional advection–diffusion equation appears mathematically simple, but its numerical solution can become challenging when the relative strengths of advection and diffusion change.

The explicit upwind scheme has an important practical advantage: its implementation is straightforward and each new value can be calculated directly from values at the previous time level. This makes it attractive for introductory computational modelling and for problems in which robustness is more important than very high accuracy. Its principal disadvantage is numerical diffusion. A sharp concentration or temperature profile may therefore become artificially broadened.

The implicit formulation reduces the severe time-step restrictions associated with explicit treatment of diffusion. Its disadvantage is that a system of algebraic equations must be solved at every time step. For a one-dimensional problem, the resulting matrix is normally sparse and structured, so efficient algorithms such as the Thomas algorithm can be employed.

Crank–Nicolson provides a useful compromise between computational efficiency and accuracy. It treats the temporal derivative using an averaging procedure and is second-order accurate in time for the standard formulation. The method has therefore been extensively investigated for transient diffusion and convection–diffusion problems.

5.2 Conclusion

The present study investigated the numerical analysis of a one-dimensional advection–diffusion transport equation using finite-difference methods. Explicit upwind, implicit and Crank–Nicolson formulations were considered from the perspectives of accuracy, stability, convergence and computational characteristics.

The numerical results indicate that finite-difference methods can successfully approximate the solution of the one-dimensional transport equation when suitable grid and time-step parameters are selected. Mesh refinement consistently reduced the numerical error. The Crank–Nicolson method produced substantially smaller errors for the smooth benchmark solution, while the upwind method demonstrated the robustness expected from a directionally biased transport discretization.

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